Definitions

Topology/Geometry of Geodesics

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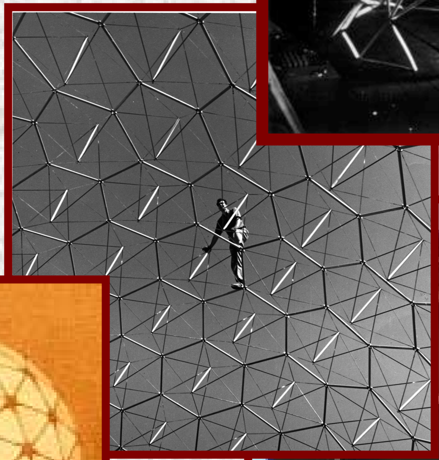
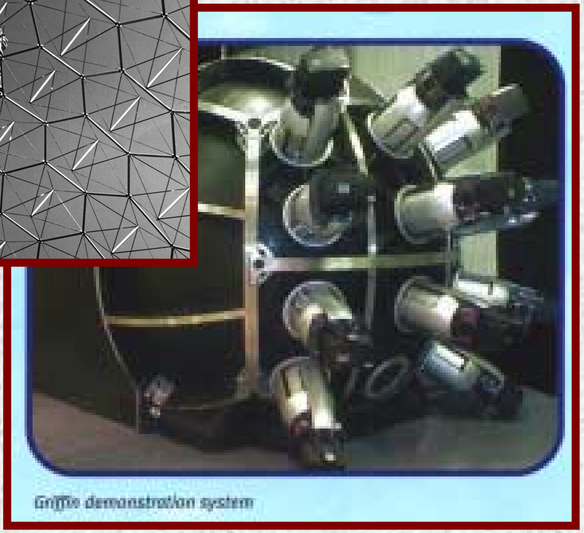
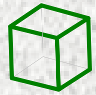
Introduction

- **Definitions**
- **Topology**
  - **Goldberg's polyhedra**
  - **Classes of Geodesic polyhedra**
  - **Triangular tessellations**
  - **Diamond tessellations**
  - **Hexagonal tessellations**
- **Geometry**
  - **Methods**







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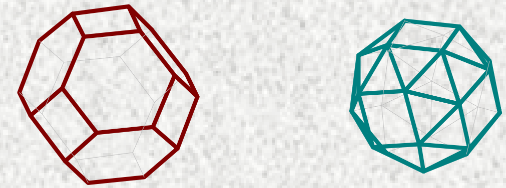
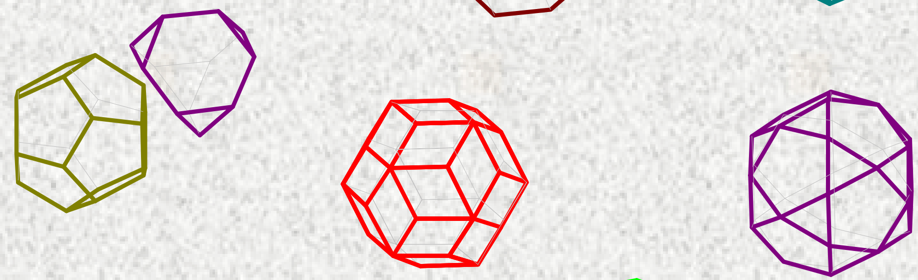
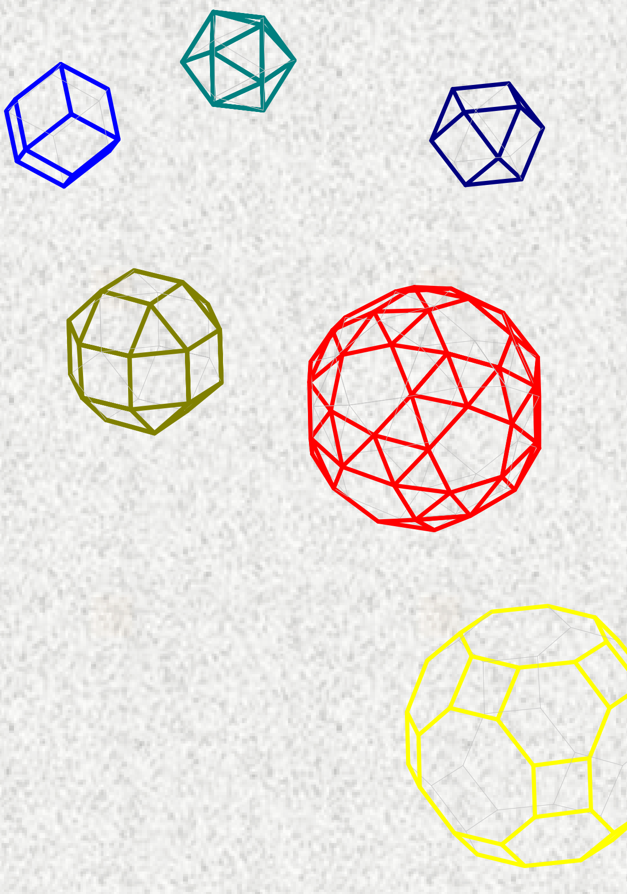
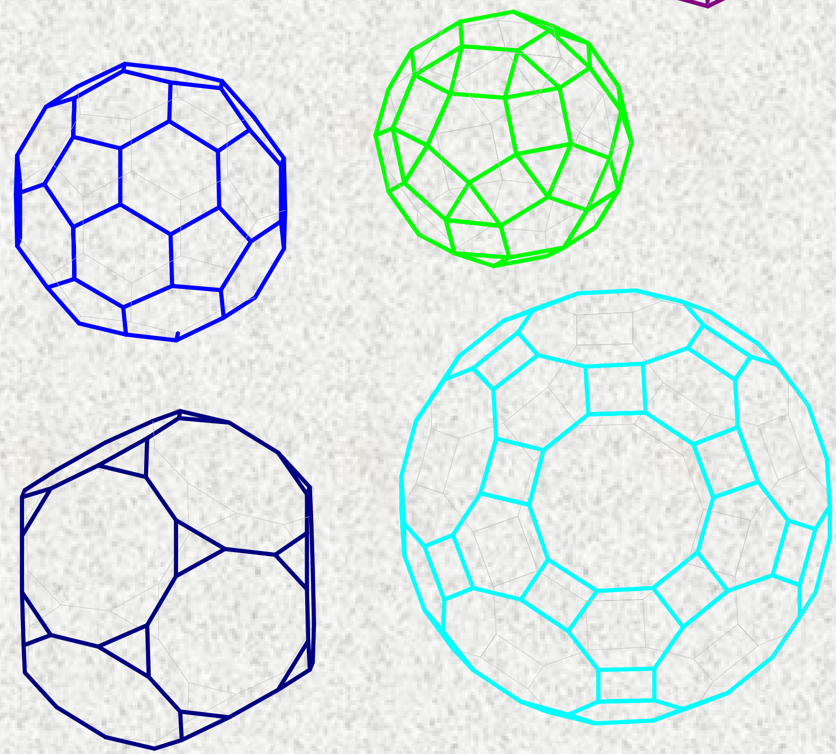


- Parent Polyhedra

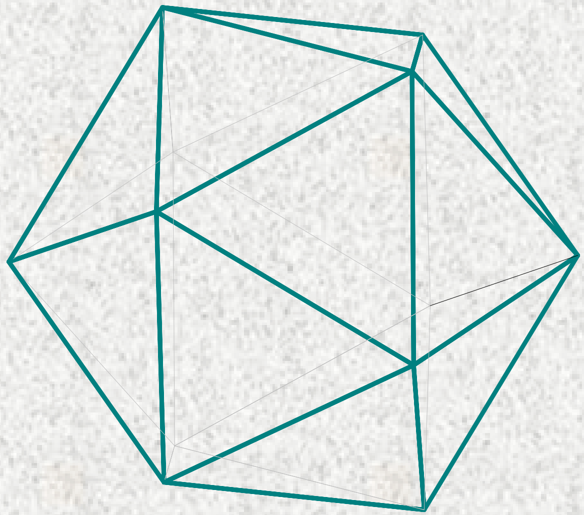


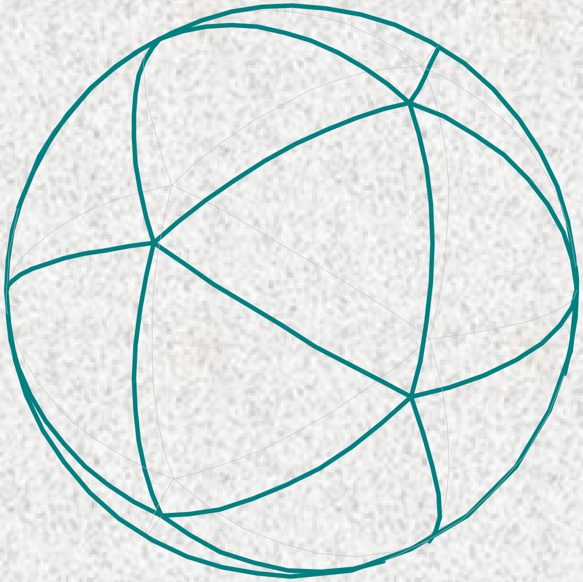




Definitions

• Planer form of the Parent Polyhedra

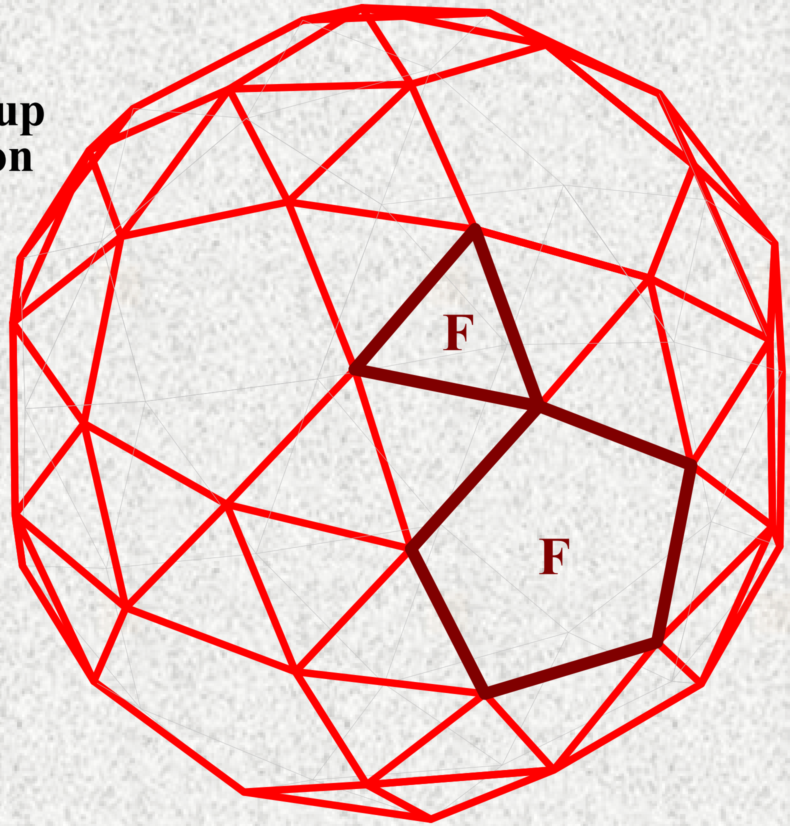




• Spherical form of the Parent Polyhedra

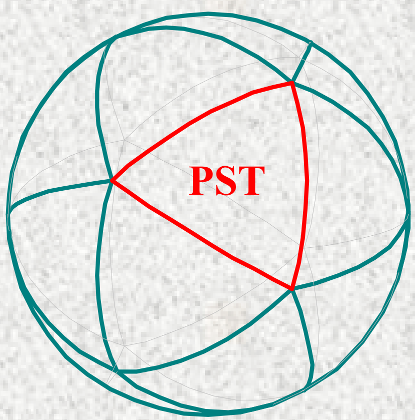
Definitions

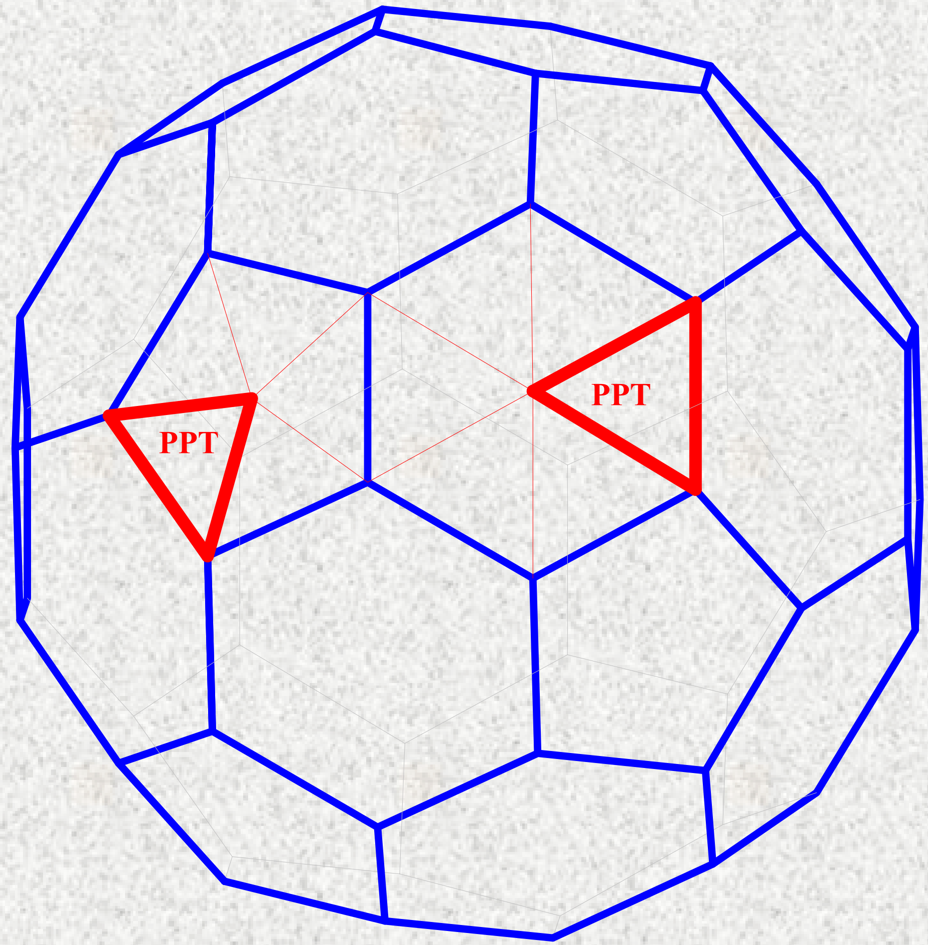
- Polyhedron Face (F)
  - Any of the plane or spherical polygons making up the surface of the polyhedron



Definitions

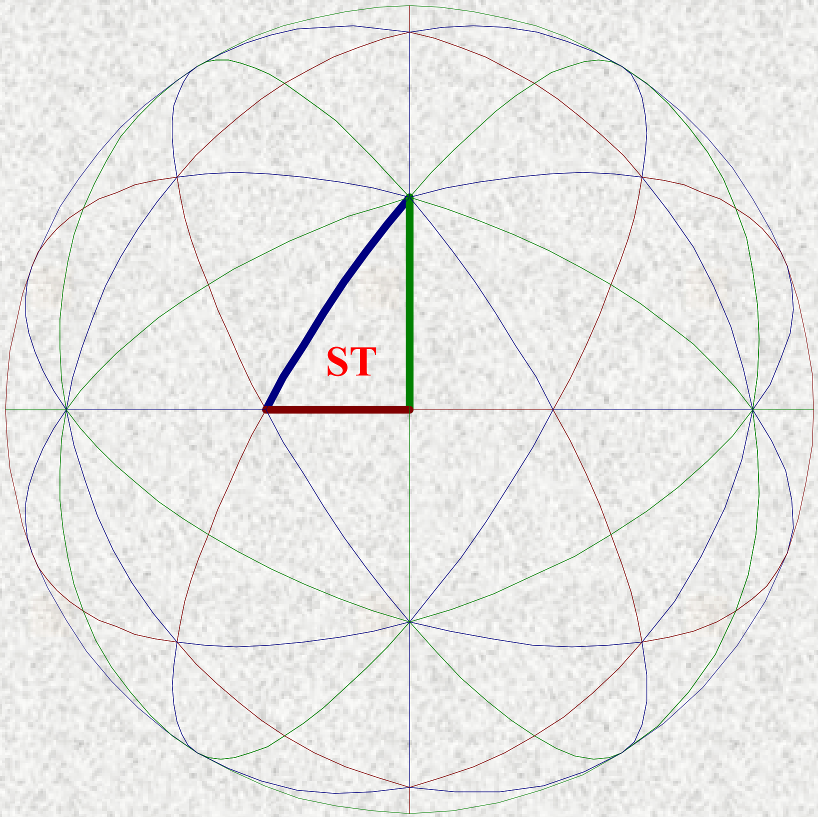
- Principle Triangle (PT)
  - One of the triangles of the parent polyhedron used in development of the three-way grid of the geodesic form.
  - It may be planer (PPT) or it may be spherical (PST)





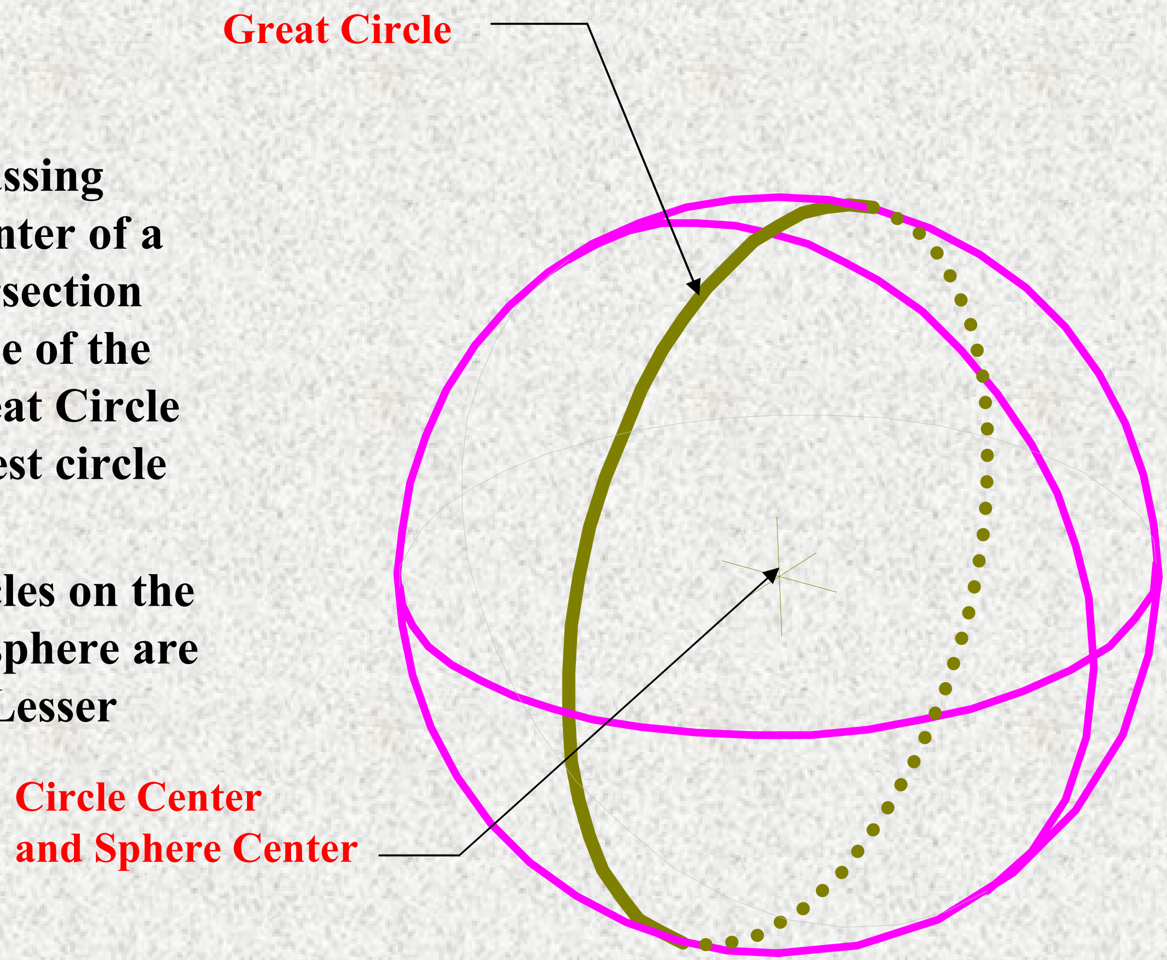
Definitions

- Schwarz Triangle (ST)
  - "A problem proposed and solved by Schwarz in 1873: to find all spherical triangles which lead, by repeated reflection in their sides, to a set of congruent triangles covering the sphere a finite number of times."
  - There are only 44 kinds of Schwarz triangles.



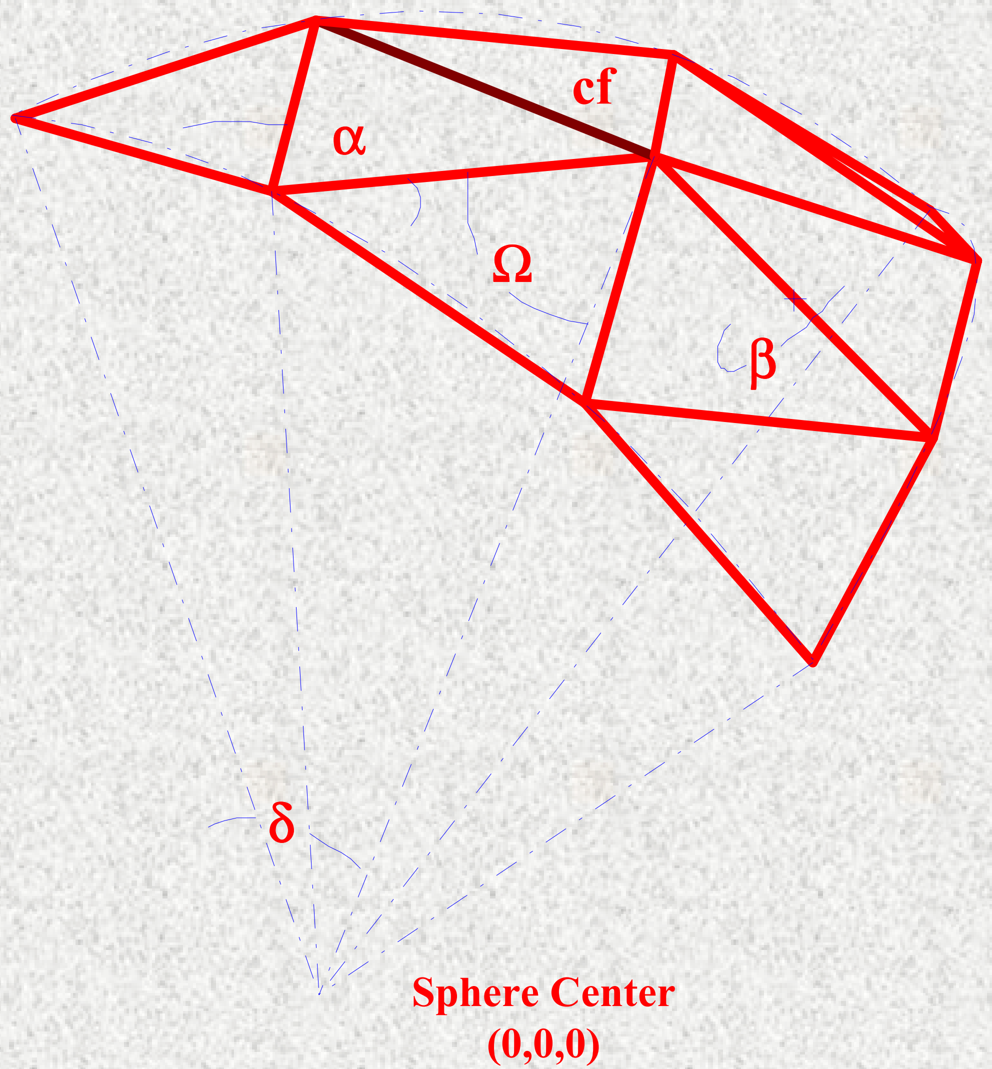
Definitions

- Great Circle
  - Any plane passing through the center of a sphere its intersection with the surface of the sphere is a Great Circle and is the largest circle on the sphere.
  - All other circles on the surface of the sphere are referred to as Lesser Circles.



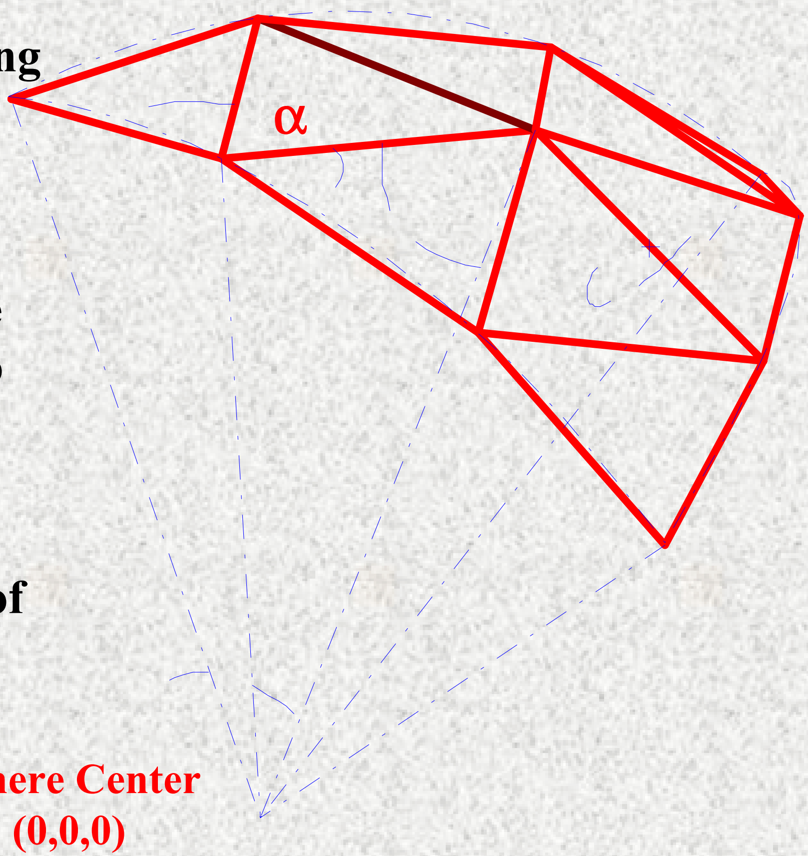
Definitions

- **Geodesic Geometrical Elements**
  - Face angle (α)
  - Dihedral angle (β)
  - Central angle (δ)
  - Axial angle (Ω)
  - Chord factor (cf)



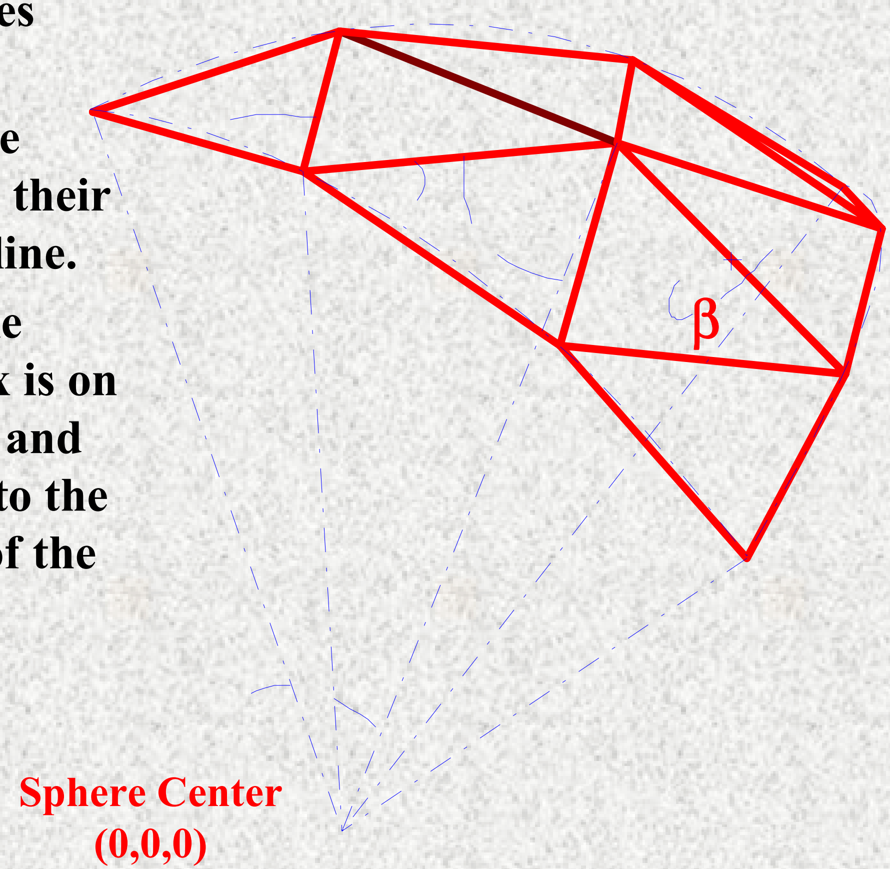
Definitions

- **Face Angle (Alpha α)**
  - – An angle formed by two chords meeting in a common point and lying in a plane that is the face of the geodesic polyhedron.
  - – For the spherical form of the face it is the dihedral angle between two great circle planes with a common intersection and containing the two chords meeting in a common point that lies on the line of intersection of the two great circle planes.



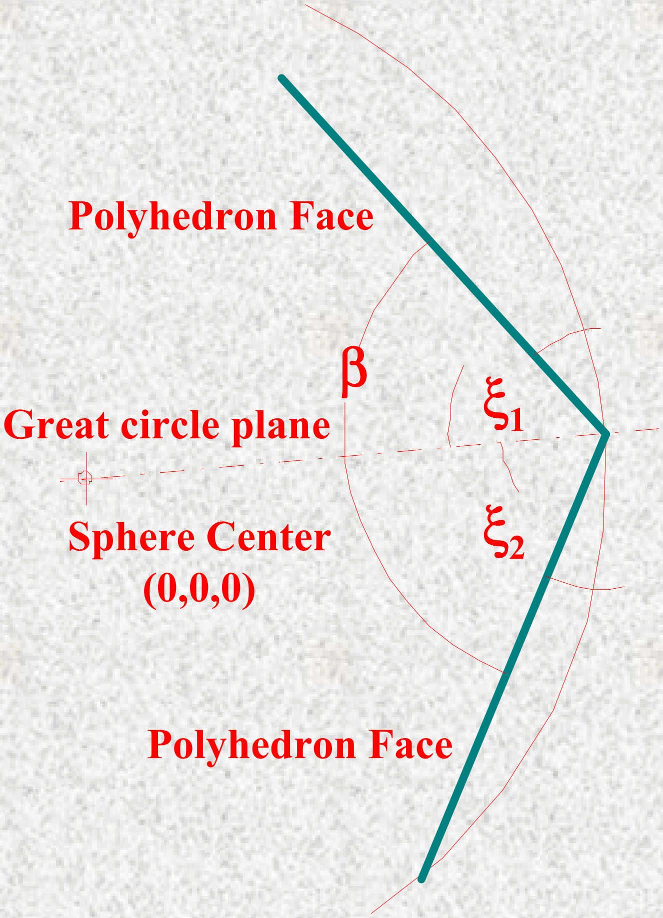
Definitions

- **Dihedral Angle (Beta β)**
  - An angle formed by two planes meeting in a common line
  - The two planes themselves are faces of the dihedral angle, and their common chord is the common line.
  - To measure the dihedral angle measure the angle whose vertex is on the chord of the dihedral angle and whose sides are perpendicular to the chord and lie one in each face of the dihedral angle.



Definitions

- Partial Dihedral Angle (Xi ξ)
  - An angle formed by two planes meeting in a common line with one plane being a face of the polyhedron and the other being the great circle plane of the chord.
  - The angle is measured the same as the dihedral angle β.
  - For each dihedral angle β there are two partial dihedral angles ξ_1 and ξ_2.

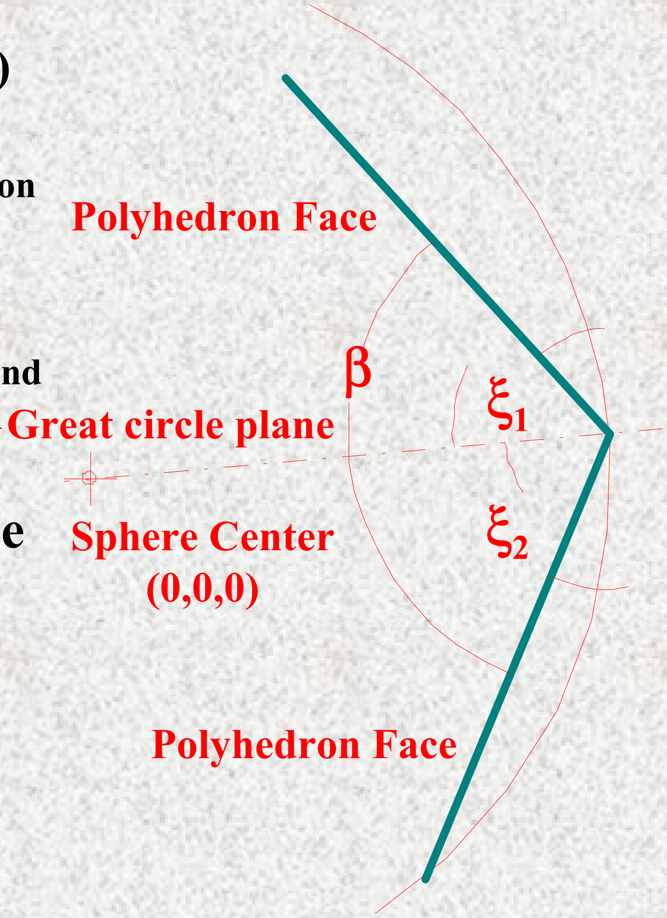


Definitions

- **Partial Dihedral Angle (Xi ξ)**

$\cos \xi_1 = (\tan \delta_1 / 2) * \tan (90 - \alpha_1)$

- **where:**
  - **δ_1 = the central angle of the common edge of the dihedral angle being measured.**
  - **α_1 = the face angle opposite the common dihedral being measured and contained in the face of the dihedral plane**

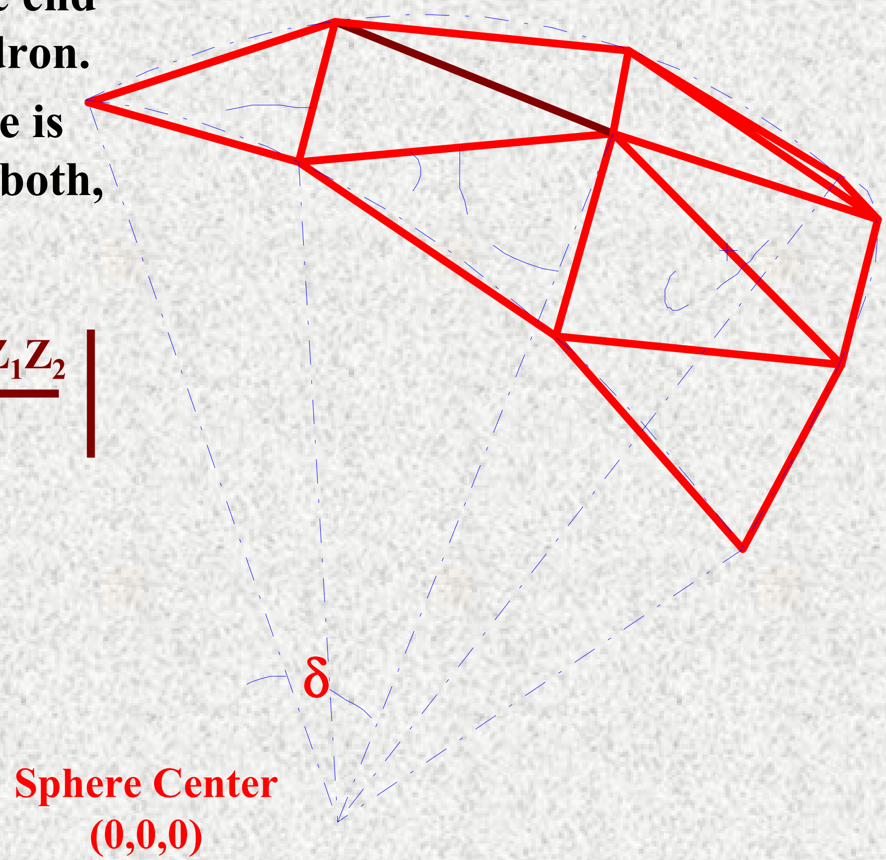


- **The dihedral angle β = the sum of the two partial dihedral angles ξ_1 and ξ_2 that share a common edge.**

Definitions

- Central Angle (Delta δ)
  - An angle formed by two radii of the polyhedron passing through the end points of a chord of the polyhedron.
  - The vertex of the central angle is chosen as the point common to both, (the center of the polyhedron).

$$\cos \delta = \left| \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{r^2} \right|$$

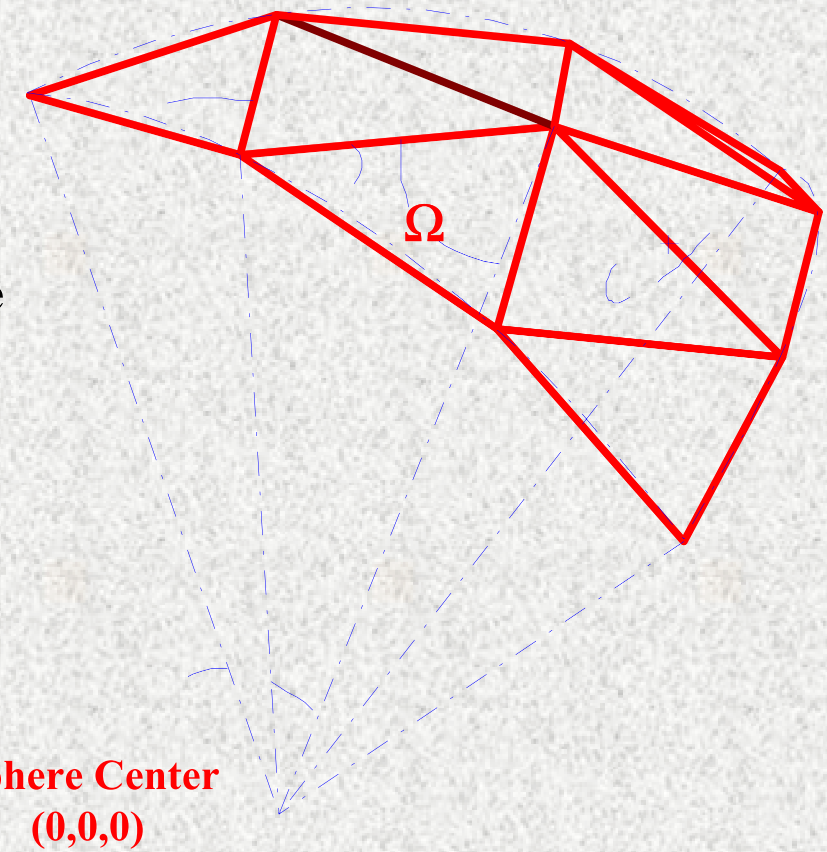


Definitions

- **Axial Angle (Omega Ω)**
  - An angle formed by a chord and the radius from the center of the polyhedron and meeting in a common point.
  - The vertex of the axial angle is chosen as the point common to the polyhedron's chord and radius.

$$\Omega = (180° - \delta) / 2$$

$$\Omega = \mathrm{ACOS}\ (cf/2)$$

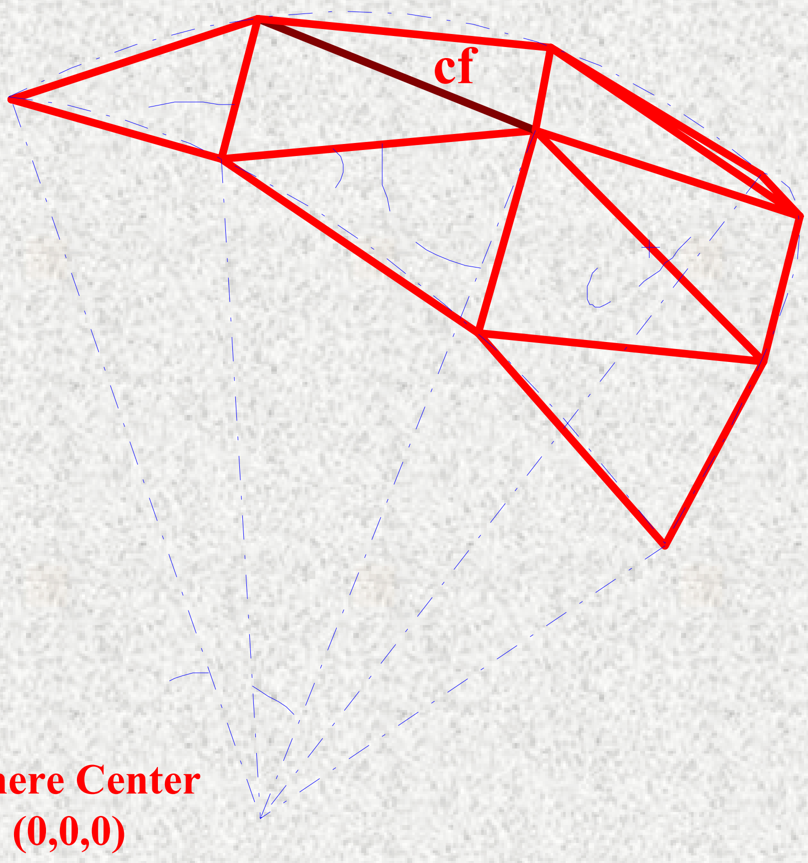


Definitions

- Chord Factor (cf)
  - The chord length of the polyhedron.
  - Based on a radius of a non-dimensional unit of one for the circumscribed sphere.
  - With the end points of the chord coincident with surface of the sphere.

    cf = 2*SIN δ/2
  - The length of the chord for any structure may be found by:

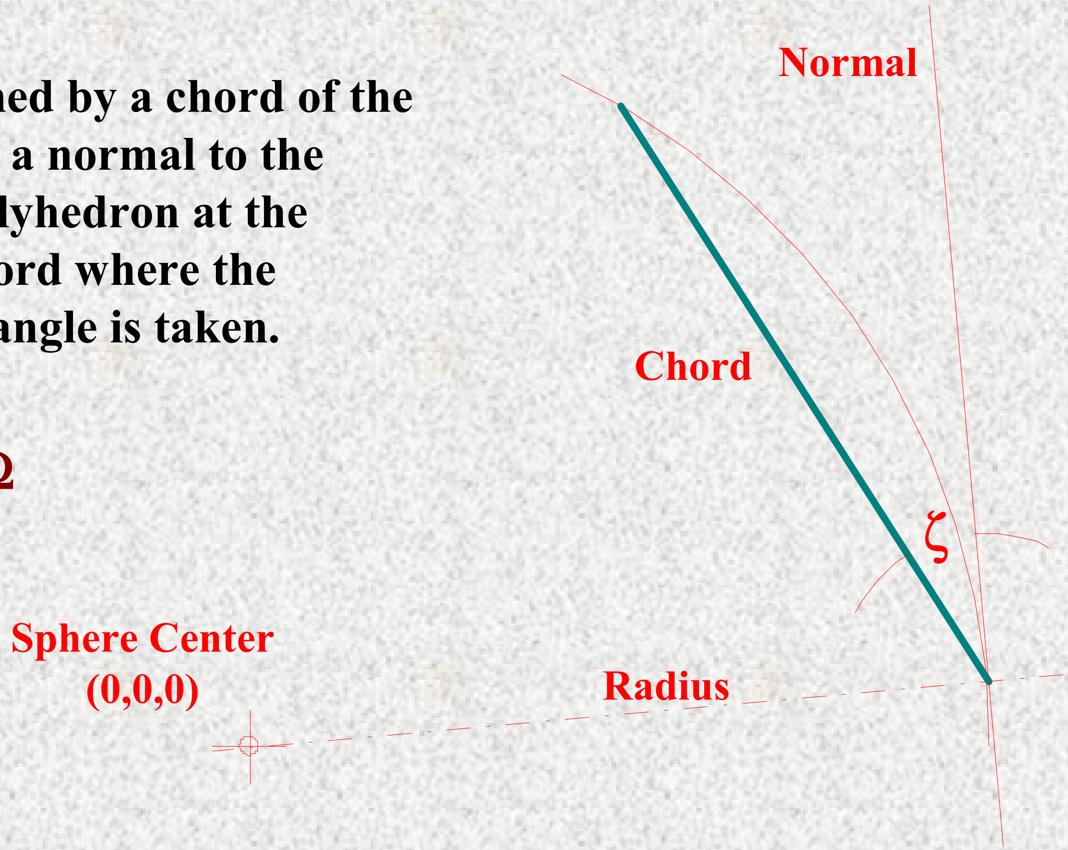
    l = cf * r where: r = radius



Definitions

- **Tangential Angle (Zeta ζ)**
  - An angle formed by a chord of the polyhedron and a normal to the radius of the polyhedron at the vertex of the chord where the measure of the angle is taken.

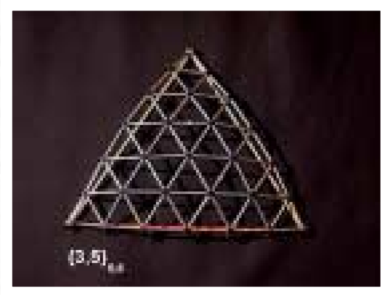
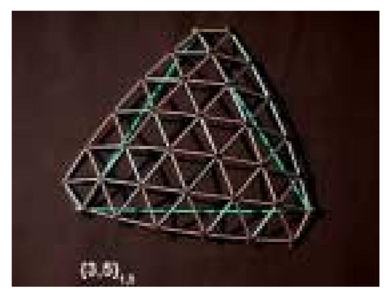
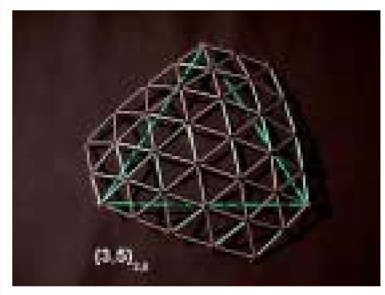
$$\zeta = 90° - \Omega$$



Definitions

- Frequency (Nu ν)
  - The number of subdivisions along a path on a PPT to get from one vertex to an adjacent vertex.
  - b and c are counters along the path.

$$\nu = b + c$$


Definitions

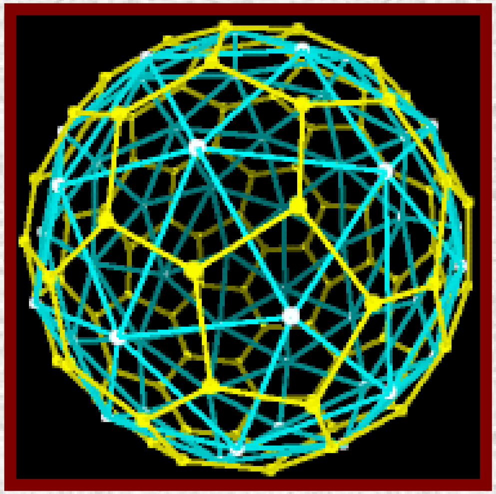
- **Quotient of LE and SE** **(Eta h)**
  - **Quotient of the longest and shortest edge of the polyhedron**
  - **Polyhedron approaching equal edges**

  Where:

$$h \leq 2 \sin 36^{o}$$

Definitions

- **Duality**
  - The vertices of a polyhedron correspond to the faces of its dual, and vice versa.
  - The edges of the two polyhedra correspond directly to one another.



Nicholas Shea

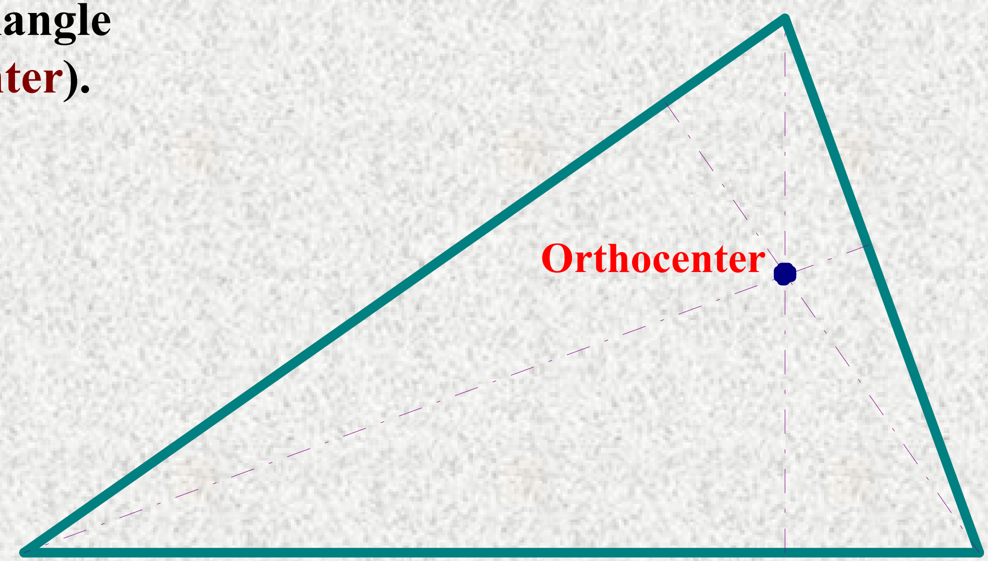
Definitions

- **Euler's theorem**
  - **The number of vertices (*v*), edges (*e*), and faces (*f*) in a spherical polyhedron are related by:**

$$v + f = e + 2$$

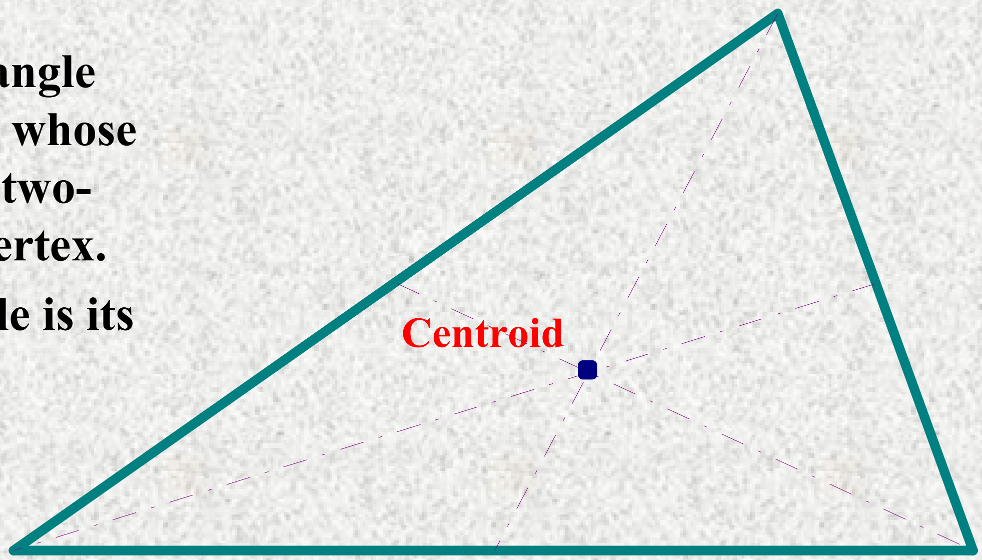
Definitions

- Altitude (h)
  - A line constructed from a vertex of a triangle ⊥ to the opposite edge.
  - The three altitudes of a triangle meet at one point (Orthocenter).



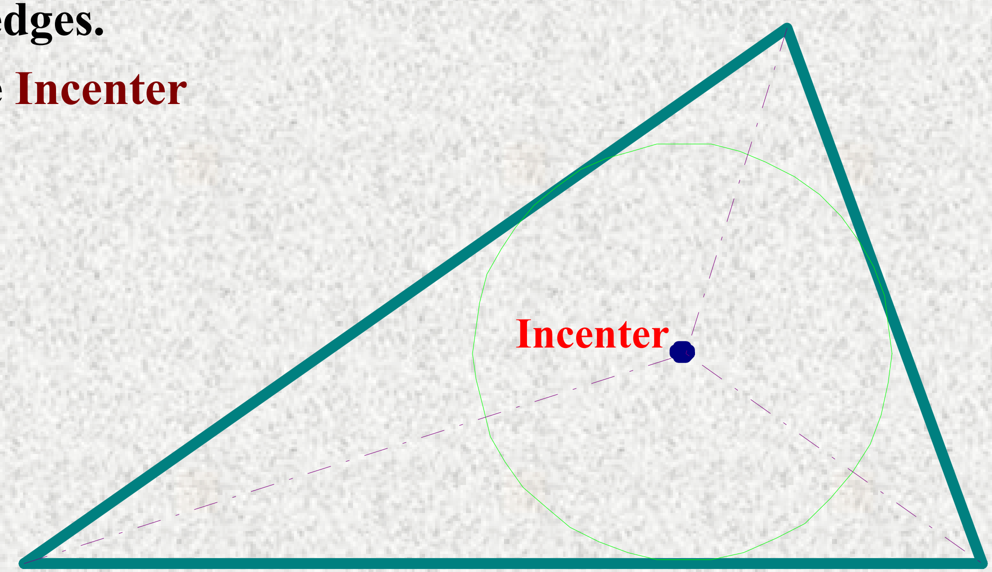
Definitions

- **Median (m)**
  - A line constructed from a vertex of a triangle to the midpoint of the opposite side.
  - The three medians of a triangle meet at one point (Centroid) whose distance from each vertex is two-thirds the median from its vertex.
  - The Centroid of the triangle is its center of mass.



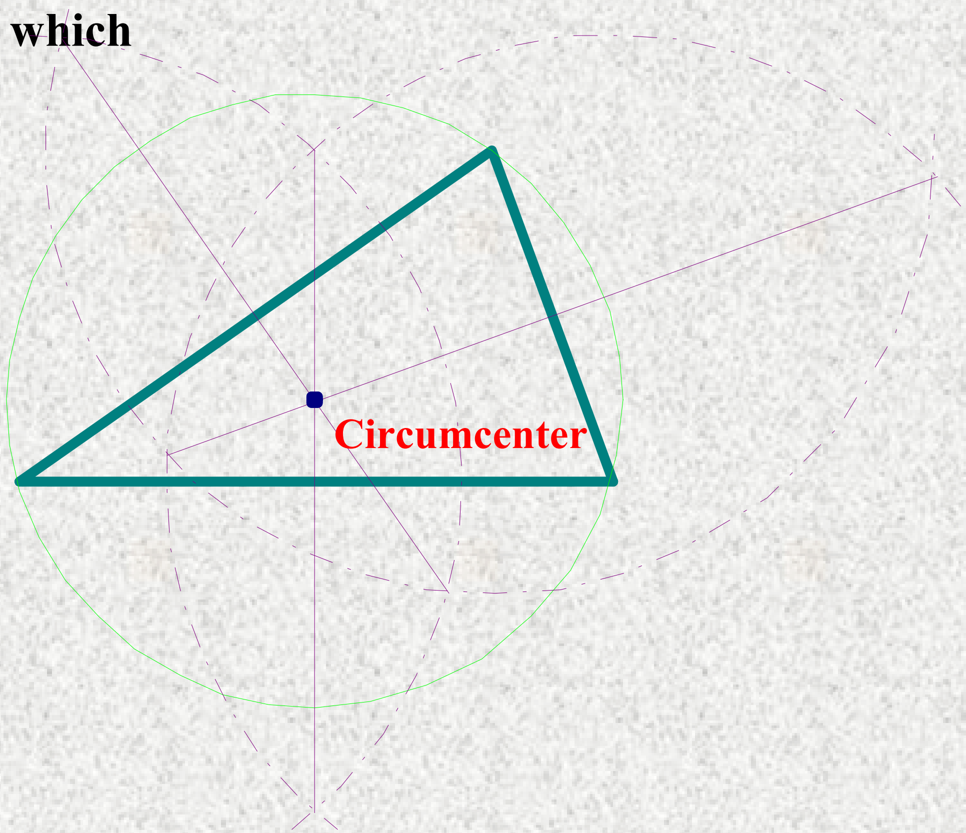
Definitions

- **Angle Bisectors**
  - **The three bisectors of a triangle meet in one point which is equidistant from the three edges.**
  - **This point is known as the Incenter**



Definitions

- **Perpendicular Bisectors**
  - **The three ⊥ bisectors of the edges of a triangle meet in one point which is equidistant from the three vertices.**
  - **This point is known as the Circumcenter.**



Definitions

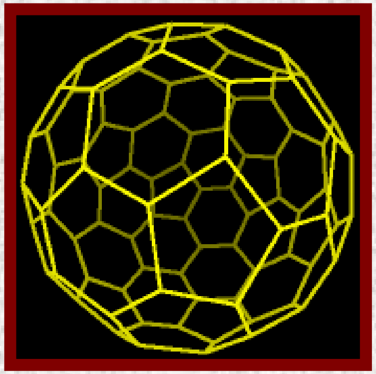
- Uniformity
- Even Distribution
  - Two terms that have many meanings

Definitions

- **Isomer (Chemistry and Biological term)**
  - **One of two or more substances which have the same elementary composition, but differ in structure, and hence in properties**

Definitions

- Voronoi Diagram
  - The partitioning of a plane with n points into n convex polygons such that each polygon contains exactly one point.
  - Every point in a given polygon is closer to its center point than any other
- Delaunay triangulation
  - The nerve of the cells in a Voronoi Diagram.
  - The triangular of the Convex Hull of the points in the diagram.



Nicholass Shea

Topology

- Topology
  - The mathematical study of properties of spatial objects where shape is preserved through deformations and size is not a consideration.
    - Goldberg's polyhedra
    - Geodesic Classes
    - Triangular Tessellation
    - Diamond Tessellation
    - Hexagonal Tessellation

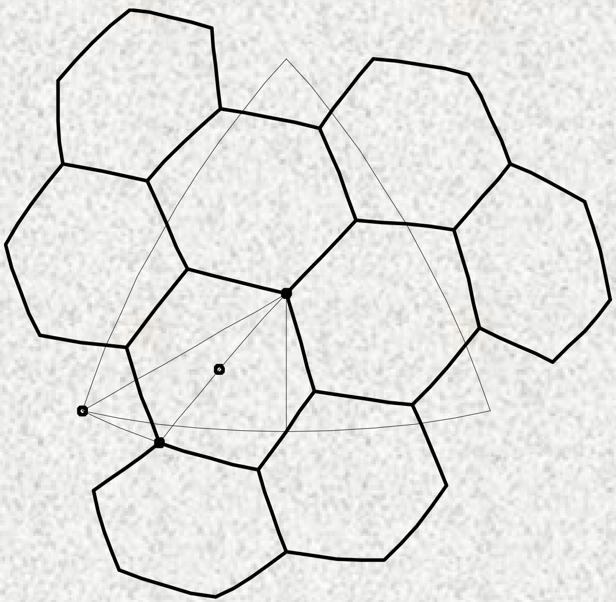
Topology

- Goldberg's polyhedra

  – "The arrangement of the hexagons in a triangular patch is the same as in a 30º sector of a regular honeycomb arrangement of hexagons…Using a, b as the inclined coordinates (60º between axes) of the vertex of a patch, the square of the distance from the center of the patch to the vertex is equal to $a^2 + ab + b^2$ …the total number of faces bounding the '*polyhedron'* is 10(a2 + ab +b2)+2;"*for the icosahedral system,* *$4(a^2 + ab + b^2)+2$ for the octahedral system, and* *$2(a^2 + ab + b^2)+2$ for the tetrahedral system.*

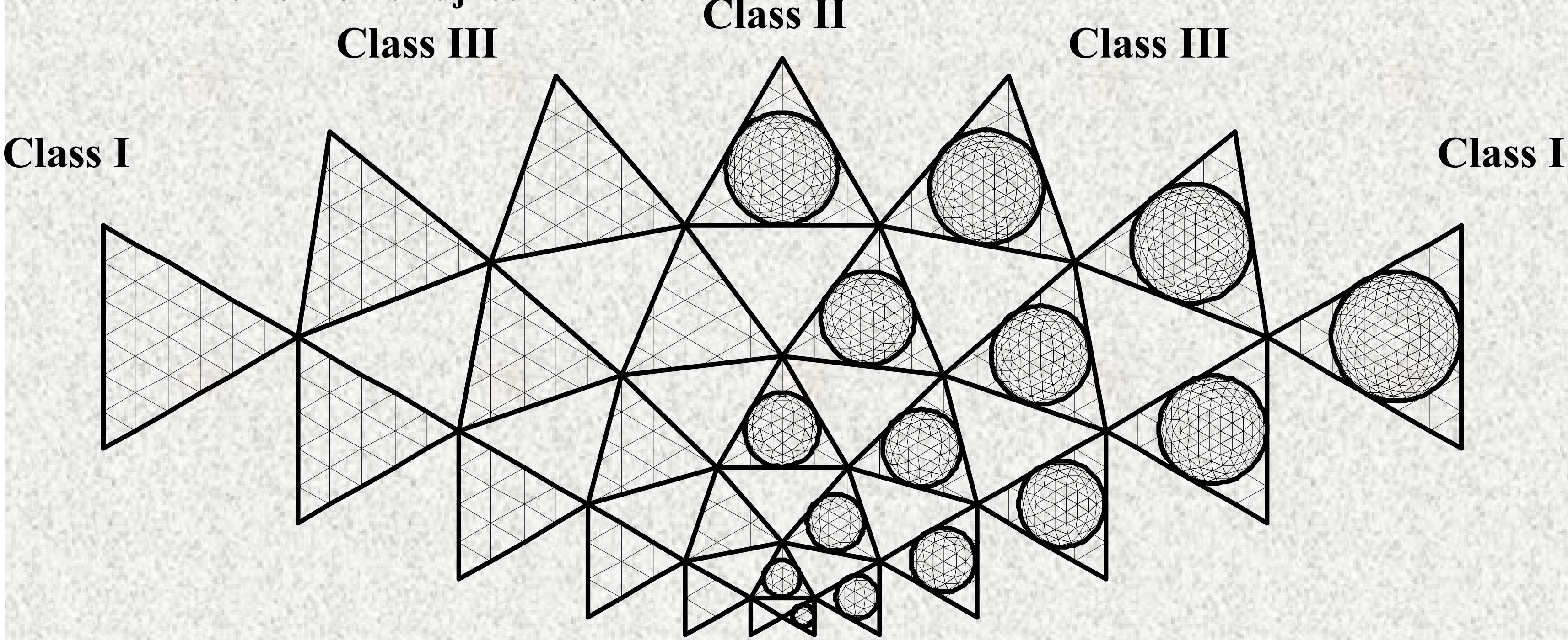
  – The a,b counters are directed along the hex/tri paths from the parent vertex to its center.

a=1, b=2



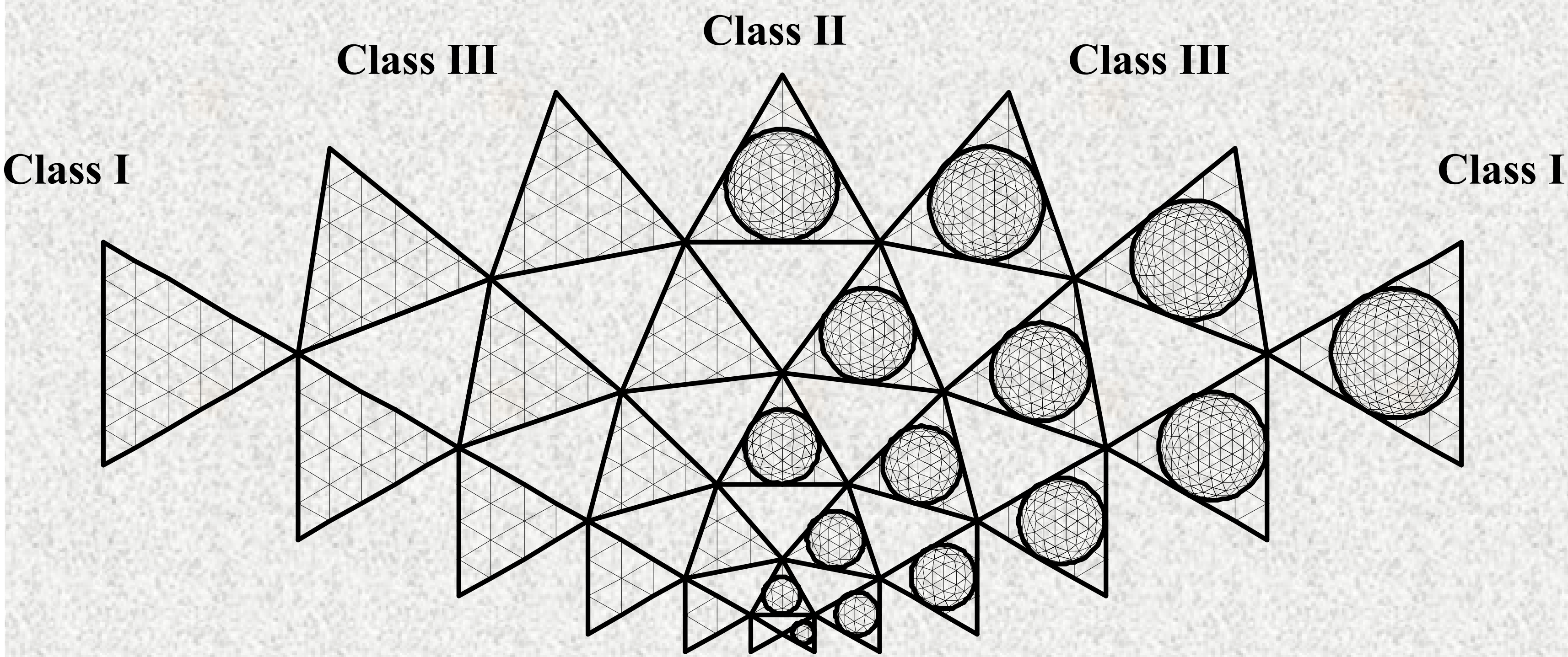
Topology

- {p,q+} $_{b,c}$ - Geodesic Classes (Clinton, Coxeter, Wenninger)
  - Class I Subscript b is an integer and c is always zero.
  - Class II Subscript b is any integer and is always equal to c.
  - Class III Subscript b and c can be any integers so long as b ≠ c
  - The b,c counters are directed along the tri paths from the parent triangle vertex to its adjacent vertex



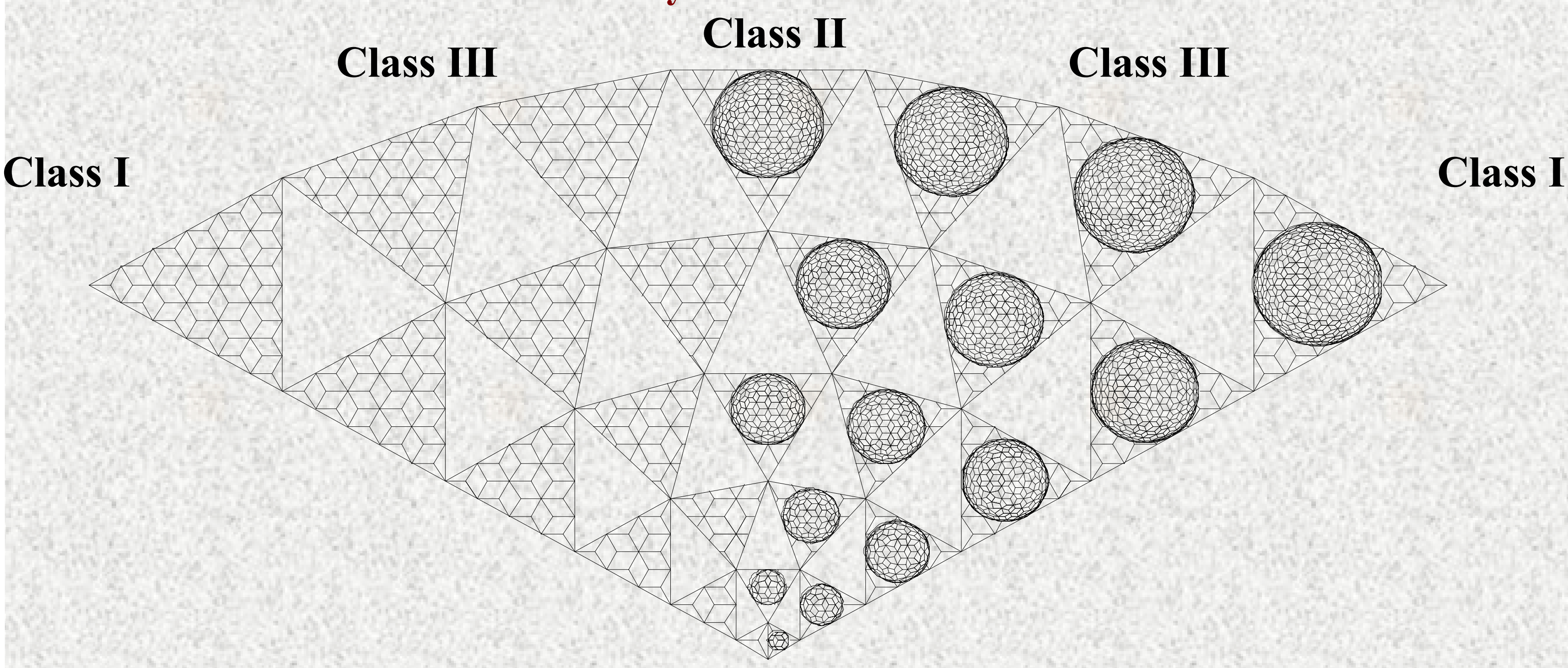
Topology

- **Triangular Tessellation**
  - **Icosahedron**
    - **F = 20T**
    - **E = 30 T**
    - **V = 10T + 2**
    - **Where: $T = b^2 + bc + c^2$**



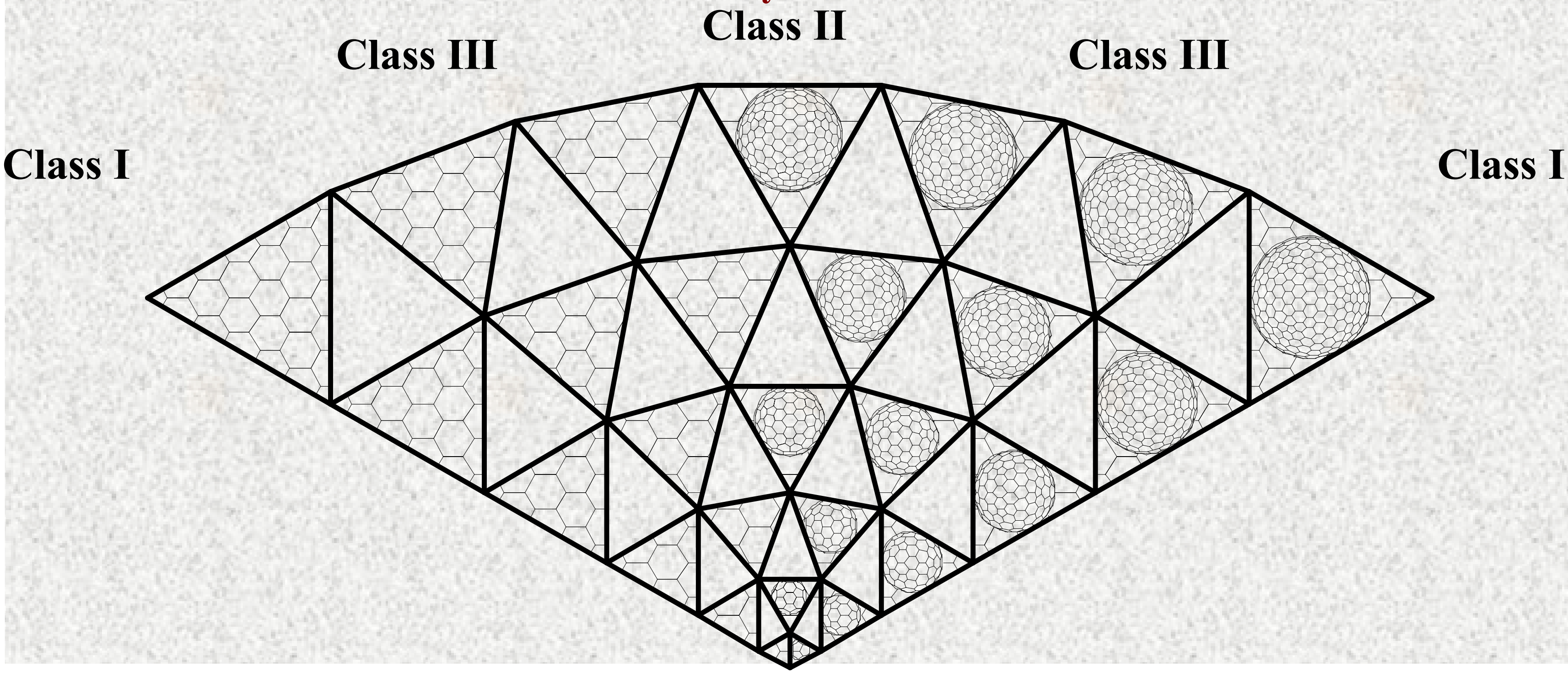
Topology

- Diamond Tessellation
  - Icosahedron
    - $F = 10T$
    - $E = 20\ T$
    - $V = 30T + 2$
    - Where: $T = b^2 + bc + c^2$
    - T must be divisible by 3



Topology

- Hexagonal Tessellation
  - Icosahedron
    - F = 10T/3 + 2
    - E = 10T
    - V = 20T/3
    - Where: $T = b^2 + bc + c^2$
    - b and c must be divisible by 3



Geometry

- Geometry
  - – The mathematical study of figures in Space of a given number of dimensions and of a given type.
    - Geodesic methods of tessellating the surface of a sphere